THE PRECESSION OF MERCURY AND THE DEFLECTION OF STARLIGHT FROM SPECIAL RELATIVITY ALONE

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ABSTRACT

When the gravitational field of a particle is treated in the same way as the electric field of a charged particle, the precession of the orbit of Mercury and the deflection of starlight by the Sun computed using special relativity agree with observation.

Subject headings: gravitation — gravitational lensing

1. The Precession of the Perihelion, or The Deflection of a "Massy" Particle

Predicting the precession of the orbit of Mercury is a standard test for relativity theories. The observed precession is 13.489"/orbit, mostly from precession of the observer's coordinate system and from interactions with other bodies in the solar system (cf. Misner, Thorne, & Wheeler 1973, p.1113). Only 0.104"±0.002"/orbit of the precession is produced by volume effects for Mercury and for the Sun and by non-radial relativistic effects (Shapiro et al. 1972). But since we do not yet know the internal structure of Mercury (or the Sun), the precession of Mercury is not a test of relativity. In fact the density distribution in Mercury can be estimated by requiring that it produce most of the observed precession.

Assume that the gravitational field produced by a massy particle varies with velocity and has the same properties as the electric field produced by a moving electric point charge that is described in electricity and magnetism textbooks. From Resnick (1968) substituting GM for $q/4\pi\epsilon_0$,

$$\vec{f} = (1-\beta^2)/[1-\beta^2 \sin^2\alpha]^{3/2}\, GM\hat{r}/r^2, \quad (1)$$

where $\beta = v/c$, v is the particle velocity, c is the speed of light, α is the angle between $\hat{v}$ and $\hat{r}$, G is the gravitational constant, M = $M_0/(1-\beta^2)^{1/2}$ is the gravitational mass, and M_0 is the rest mass. The field moves away from the poles, the direction of motion, toward the equator as the velocity increases. The $\alpha\beta$ factor is $(1-\beta^2)$ at the poles which goes to 0 as v approaches c. The $\alpha\beta$ factor is $1/(1-\beta^2)^{1/2}$ at the equator which becomes large as v approaches c. Substituting $\sin^2\alpha = 1-\cos^2\alpha = 1-(\hat{v}\cdot\hat{r})^2$,

$$\vec{f} = (1-\beta^2)/[1-\beta^2(1-(\hat{v}\cdot\hat{r})^2]^{3/2}\, GM\hat{r}/r^2. \quad (2)$$

The integral of the force over the volumes of Mercury and the Sun does not degenerate into a two-body problem but it can be treated as a two-body problem in heliocentric coordinates with perturbations. The subscripts S and M refer to the Sun and Mercury.

$$\vec{F} = -GM_M(M_\odot + M_M)(1 - v^2/c^2)/[1 - v^2/c^2(1 - (\hat{v}\cdot\hat{r})^2]^{3/2}\ \hat{r}/r^2. \quad (3)$$

where $\hat{r}$ and $\hat{v}$ are the position and velocity of the center of mass of Mercury relative to the position and velocity of the center of mass of the Sun, which are defined to be 0. The two-body force is purely radial. The acceleration toward the Sun is $\vec{a} = \vec{F}/M_M$. From the calculations described below, typical values in this problem are: period 88 days; r = 0.387 AU or 58 million km; v = 48 km/s; β = 0.00016; $1 - \beta^2$ = 0.999999974; $M_{0M} = 3.3\times10^{26}$ g; $M_{0M}/M_\odot$ = 0.000000166; M_M = $1.0000000128\times3.3\times10^{26}$ g; $\hat{v}\cdot\hat{r}$ = -0.20 to +0.20. The precession is about 29 km/orbit.

The perturbative forces are defined at the same time and positions as the two-body force, at the center of mass of the Sun and the center of mass of Mercury. Define $\vec{r}_S$ and $\vec{v}_S$ as the position and velocity vectors of a mass element in the Sun relative to the center of the Sun and $\vec{r}_M$ and $\vec{v}_M$ are the position and velocity vectors of a mass element in Mercury relative to the center of mass of Mercury. Let $\vec{d} = \vec{r} + \vec{r}_S + \vec{r}_M$ and $\vec{w} = \vec{v} + \vec{v}_S + \vec{v}_M$. Mercury and the Sun are far apart so they present small solid angles to each other. Assume that the Sun is spherical with density that decreases radially from 148 to 0 g/cm^3 (Lebretton & Dappen 1988). Assume that the Sun rotates with a surface equitorial velocity of about 2 km/s and that internal motions are smaller than 2 km/s and symmetric about the equator. In the solar part of the integrand of the force, angular effects and retardation effects are small (which I have tested by numerical integration as in Section 2) so that $\vec{r}_S$ and $\vec{v}_S$ can be ignored and the solar part of the integrand can be factored out. Assume that Mercury is spherical with density that decreases radially from about 9 to about 3 g/cm^3 (Schubert et al. 1988). A mostly iron core fills about three-fourths of the 2440 km radius and a rocky mantle is the outer fourth. The rotation of Mercury is small and internal motions are small or non-existent so $\vec{v}_M$ is small and $\vec{w} = \vec{v}$. The force reduces to

$$\vec{F} = -GM_M(M_\odot + M_M)\int_M (1 - v^2/c^2)/[1 - (v^2/c^2)(1 - (\hat{v}\cdot\hat{d})^2]^{3/2}\rho_M\ \hat{d}/d^2 dV_M/M_M, \quad (4)$$

where $\vec{d} = \vec{r} + \vec{r}_M$ and where all the mass points are retarded to the center of mass of Mercury. The retardation in time is $dt = -(d - r)/c$, and in position is $\vec{d}\,' = \vec{d} - \vec{v}(d - r)/c$. This force has non-radial components. Since β is small the denominator can be expanded to yield

$$\vec{F} = -GM_M(M_\odot + M_M)\int_M [1 + 1/2\beta^2 - 3/2\beta^2(\hat{v}\cdot\hat{d})^2]\ \rho_M\ \hat{d}/d^2\ dV_M/M_M. \quad (5)$$

I approximately computed the precession as follows: First the orbit was computed as a two-body problem with [3] using Butcher's 5th order Runge-Kutta method following Boulet (1991) for 50000 steps/day. The perihelion was determined by 6-point Lagrangian differentiation. The two-body orbit repeated the perihelion to 100 μarcsec. A vectorial correction factor to the two-body force for the volume of Mercury was tabulated five times per day for that orbit by integrating

[5] over 500 density shells, 500 latitudes, and 1000 longitudes for a range of Mercury models that tabulate density as a function of radius. The precession for a uniform density of 5.426 g/cm^3 was 0.117"/orbit. An iron-core-with-thin-mantle model that I made up inspired by Figure 1 in Schubert et al. (1988) yielded a precession of 0.107"/orbit. Thus, using only special relativity, a simple model reproduces the precession to within 3 percent. Keeping the thickness of the mantle about one fourth the radius, I tried ad hoc fixes to the density variation in the model until a precession of 0.105"/orbit was achieved in agreement with observation. Several different models produced the same result. Remember that other small volume effects in Mercury and the Sun were ignored in this calculation; neither is actually spherical. Seismic measurements on Mercury itself will eventually allow a real model to be determined that will test the relativistic calculation.

2. The Deflection of Starlight, or The Deflection of a Massless Particle

The gravitational field expands from a particle at the speed of light. But a massless particle moves at the speed of light. In the direction of motion there can be no gravitational field. In analogy with the relativistic contraction of the field of a massy particle described above, the whole gravitational field of a massless particle is in the plane perpendicular to the direction of motion. Instead of filling a solid angle 4π, the field contracts to the "width" of the equator $d\phi$ with solid angle $2\pi d\phi$ and "strength" S. Then $S2\pi d\phi = 4\pi$ and $Sd\phi = 2\delta_\perp$ where $\delta_\perp$ is a δ-function in the equatorial plane. The gravitational mass of a photon, or other massless particle is $2\delta_\perp \mathrm{E/c}^2$. The gravitational mass averaged over all directions is E/c^2 and the momentum of a massless particle is $(E/c)\hat{c}$.

A massless particle gains energy, blueshifts, as it approaches a massive body B and loses energy, redshifts, as it departs. The gain in energy is also a gain in mass. The gravitational mass M_E is $E/c^2 = (1 + GM_B/r/c^2)E_{r=\infty}$. The two-body force on a massless particle passing a massive body B in B-centric coordinates is

$$\vec{F} = -GM_BM_E\ 2\sin\alpha\ \hat{p}/r^2 = -GM_BM_E\ 2\sqrt{1-(\hat{c}\cdot\hat{r})^2}\ \hat{p}/r^2, \tag{6}$$

where $\hat{r}$ is the vector from the center of mass of B to the massless particle, α is the angle between $\hat{r}$ and $\hat{c}$, and $\hat{p}$ is perpendicular to $\vec{c}$, $\hat{c}\cdot\hat{p} = 0$, and lies in the plane defined by $\vec{c}$ and $\vec{r}$. The acceleration is $\vec{a} = \vec{F}/M_E$. The general case for an extended body B with internal motion is

$$\vec{F} = -GM_E\ 2\int_B \rho\sqrt{1-(\hat{c}\cdot\hat{d})^2}\ \hat{p}/d^2\ dV, \tag{7}$$

where $\vec{d} = \vec{r} + \vec{r}_B$ and the mass points are retarded to the center of mass of B, $\hat{r}_B$ and $\vec{v}_B$ are the position and velocity vectors of a mass element in B relative to the position and velocity of the center of mass of B which are defined to be 0. Here $\hat{p}$ is perpendicular to $\vec{c}$, $\hat{c}\cdot\hat{p} = 0$, and lies in the plane defined by $\vec{c}$ and $\vec{d}$. The retardation in time is $dt = -(d-r)/c$, and in position is $\vec{d}\,' = \vec{d} - \vec{v}_B(d-r)/c$.

There are no observations of deflection by point masses. The observations are of lensing by ill-defined bodies far away and lensing by the Sun. The solar observations are that photons from a distant star or planet are deflected by 1.751±0.002 $R_{\odot}/R_{min}$ arcsec (Robertson, Carter, & Dillinger 1991) as they pass the Sun and are observed at the earth, where R_{min} is the closest approach of the photon to the center of the Sun.

I approximately computed the (half) deflection as follows: The orbit was computed as a two-body problem with [7] using Butcher's 5th order Runge-Kutta method following Boulet (1991) with the constraint that $v = c$. The path ran for 150 million km in the tangential direction starting from R_{min} 700000 km. The time step was 0.005 s near the Sun and increased outward. The Sun was modelled as a sphere with no internal motions, so no retardation. The radial density distribution was interpolated from Lebretton & Dappen (1988), normalized to the solar mass, and the force was integrated over 500 density shells, 500 latitudes, and 1000 longitudes. I integrated the force over the volume at each step and corrected the two-body force. The deflection was found to be 1.751 $R_{\odot}/R_{min}$ arcseconds in agreement with observation. Actually the same deflection is found by treating the Sun as a point mass, but the force is slightly weaker near the Sun, $t < 1.866$ s, and slightly stronger further away. Thus the deflection of photons is an effect of special relativity.

Equation [6] predicts that a photon moving radially away from a point mass experiences no force and is not gravitationally redshifted, for when $(\hat{c} \cdot \hat{r})^2 = 1$, $\vec{F} = 0$. In Newtonian gravity a sphere with a radial density distribution produces the same force as a point mass at its center. This is a mathematical accident, not physics. Real physical objects are not point masses and cannot be treated as point masses. A photon at the surface of a sphere sees a whole 2π hemisphere, not a point.

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